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B.Sc. Sem 4 – MIC

(1)

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Que :– (1) State and prove Kummer's Test.

Theorem :– Suppose that $u_n > 0$, $v_n > 0$ and that $\sum v_n$ is properly divergent. Let $\lim\left(\frac{1}{v_n} \cdot \frac{u_n}{u_{n+1}} - \frac{1}{v_{n+1}}\right) = K$. Then the series $\sum u_n$ is convergent if $K > 0$ and is divergent if $K < 0$.

Proof :– Since $\lim\left(\frac{1}{v_n} \cdot \frac{u_n}{u_{n+1}} - \frac{1}{v_{n+1}}\right) = K$

therefore, we can find $n \geq m$ such that

$$K - \epsilon < \frac{1}{v_n} \cdot \frac{u_n}{u_{n+1}} - \frac{1}{v_{n+1}} < K + \epsilon \quad \text{whenever } n \geq m$$

(a) Suppose first that $K > 0$

then from the first part of inequality by taking $\epsilon = \frac{K}{2}$. We can find a number m such that for all $n \geq m$

$$\frac{1}{v_n} \cdot \frac{u_n}{u_{n+1}} - \frac{1}{v_{n+1}} > \frac{1}{2} K$$

Since u_{n+1} is positive, multiplying throughout by u_{n+1}. We get

$$\frac{u_n}{v_n} - \frac{u_{n+1}}{v_{n+1}} > \frac{1}{2} K \cdot u_{n+1}$$

i.e $u_{n+1} < \frac{2}{K}\left(\frac{u_n}{v_n} - \frac{u_{n+1}}{v_{n+1}}\right)$ in succession and adding

putting $n = m, m+1, m+2, \ldots n-1$

We get $\sum_{n=m+1}^{n} u_r < \frac{2}{K}\left(\frac{u_m}{v_m} - \frac{u_n}{v_n}\right) < \frac{2}{K} \cdot \frac{u_m}{v_m}$

Hence $S_n = \sum_1^n u_r = \sum_1^m u_r + \sum_{m+1}^{n} u_r < \sum_1^m u_r + \frac{2}{K} \cdot \frac{u_m}{v_m}$

But since m is a fixed number the R.H.S a fixed and definite. Denote its value by λ. Then for all n, $S_n < \lambda$

Now S_n is a series consisting of n terms which are all positive and hence $\{S_n\}$ is a monotonic increasing sequence.

Also $S_n < \lambda$ for all n.

i.e S_n is bounded above. Hence it converges to a limit i.e $S_n \to S \le \lambda$ and Σu_n is Convergent

(b) Next suppose that $K < 0$

Then from the second part of the inequality by taking $\epsilon = -K (>0)$

we can find a number m such that for all $n \geqslant m$

$$\frac{1}{v_n} \frac{u_n}{u_{n+1}} - \frac{1}{v_{n+1}} \le 0$$

$$\Rightarrow \frac{u_n}{v_n} - \frac{u_{n+1}}{v_{n+1}} \le 0$$

$$\Rightarrow \frac{u_{n+1}}{v_{n+1}} > \frac{u_n}{v_n} \qquad \text{———} \quad (1)$$

$$\Rightarrow u_{n+1} \geqslant \frac{u_n}{v_n} \cdot v_{n+1}$$

Putting $n = m, m+1, m+2, \ldots, n-1$ successively

we get
$$u_{m+1} \geqslant \frac{u_m}{v_m} \cdot v_{m+1}$$

$$u_{m+2} \geqslant \frac{u_{m+1}}{v_{m+1}} \cdot v_{m+2}$$

$$\geqslant \frac{u_m}{v_m} \cdot v_{m+2} \quad \text{From (1)}$$

$$u_n > \frac{u_m}{v_m} \cdot v_n$$

Adding
$$\sum_{r=m+1}^{n} u_r \geqslant \frac{u_m}{v_m} (v_{m+1} + v_{m+2} + \cdots + v_n) \geqslant \frac{u_m}{v_m} \sum_{r=m+1}^{n} v_r$$

Now since Σv_n is divergent therefore by the Comparison test the series

Σu_n is divergent.